Explicit moduli spaces of abelian varieties with automorphisms

Bert van Geemen (joint work with Matthias Schütt)

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Outline

Introduction

Example of a **Shimura variety**:
a moduli space of ppav's with an automorphism,
i.e. of triples (X, L, ϕ):

- X a complex torus ($X \cong V/\Gamma$),
- L ample line bundle on X, which gives a principal polarization (equiv: $h^0(L) = 1$),
- ϕ is an automorphism of (X, L):

$$\phi : X \xrightarrow{\cong} X, \quad \phi(0) = 0, \qquad \phi^* L \sim L.$$

$A_{g,*}$: Moduli space of ppav's with level structure $*$
(for example, $* =$ level n: $\alpha : A[n] \xrightarrow{\cong} (\mathbb{Z}/n\mathbb{Z})^{2g}$)
which is a Galois cover with group G of A_g:

$$A_{g,*} \longrightarrow A_g = A_{g,*}/G.$$

Shimura variety as fixed point set

Given (X, L, ϕ) and a point $[(X, L, \alpha)] \in A_{g,*}$ then

- define $\phi^*[(X, L, \alpha)] = [(X, L, \alpha \circ \phi)]$,
- you get $\phi^* \in G$, (more precisely: $\alpha \circ \phi^* \circ \alpha^{-1} \in G$)
- $[(X, L, \alpha)] = [(X, L, \alpha \circ \phi)]$ (isomorphic objects), so $[(X, L, \alpha)]$ is a *fixed point* for $\phi^* \in G$ in $A_{g,*}$

Hence: moduli space of triples (X, L, ϕ),
with level structure $*$,
is the fixed point locus $(A_{g,*})^{\phi^*}$, a **Shimura variety**.

G-equivariant map

Given a G-equivariant embedding

$$\Theta : A_{g,*} \longrightarrow \mathbb{P}^N, \qquad \Theta \circ g = M_g \circ \Theta,$$

for $g \in G$, $M_g \in \mathrm{Aut}(\mathbb{P}^N)$,

the image of the moduli space of triples (X, L, ϕ) with level structure $*$ is

$$\Theta((A_{g,*})^{\phi^*}) = \Theta(A_{g,*}) \cap \mathbb{P}_\lambda$$

where $\mathbb{P}_\lambda$ is an eigenspace of M_g.

Outline

To do:

- specify the triples (A, L, ϕ),
- specify level structure $*$,
- find G-equivariant map $\Theta : A_{g,*} \longrightarrow \mathbb{P}^N$,
- determine $M_{\phi^*} \in \mathrm{Aut}(\mathbb{P}^N)$ and its eigenspaces $\mathbb{P}_\lambda$,
- find equations for $\Theta(A_{g,*})$,
- study the intersection $\Theta(A_{g,*}) \cap \mathbb{P}_\lambda$.
- Applications to Arithmetic and Geometry of Shimura varieties

The Abelian varieties

$$(B_0, L_0) \;:=\; Jac(C), \qquad C: \quad y^2 = x^5 + 1$$

C is a genus 2 curve, B_0 is a ppav with automorphism

$$\phi \;:\; B_0 \;\longrightarrow\; B_0, \qquad \phi = \phi_C^*, \quad \phi_C(x,y) = (\zeta x, y)$$

where ζ is a primitve 5-th root of unity ((B_0, L_0, ϕ) is unique). (B_0, L_0, ϕ) is rigid. Consider the 4 dim ppav with automorphism

$$(A_0, L, \phi_k) \;:=\; (B_0 \times B_0, L_0 \boxtimes L_0, \phi \times \phi^k).$$

Deformation space has dimension:

$$\dim\,(\text{Deformations } (A_0, L, \phi_k)) \;=\; \begin{cases} 1 & k = 2, 3, \\ 2 & k = 4. \end{cases}$$

The level structure $*$=(2,4)

Symmetric theta structure of level two, $(2,4)$.

$$A_{g,4} \longrightarrow \underbrace{\underbrace{A_{g,(2,4)} \longrightarrow A_{g,2}}_{\text{group } (\mathbb{Z}/2\mathbb{Z})^{2g}} \longrightarrow A_g}_{\text{group } G}.$$

There is a non-split exact sequence:

$$0 \longrightarrow (\mathbb{Z}/2\mathbb{Z})^{2g} \longrightarrow G \longrightarrow Sp(2g,\mathbb{F}_2) \longrightarrow 0.$$

$Sp(2g,\mathbb{F}_2)$ is generated by **transvections**: for $v \in \mathbb{F}_2^{2g}$

$$t_v : \mathbb{F}_2^{2g} \longrightarrow \mathbb{F}_2^{2g}, \qquad w \longmapsto w + E(w,v)v,$$

$E : \mathbb{F}_2^{2g} \times \mathbb{F}_2^{2g} \to \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ is the symplectic form.

The G-equivariant map $\Theta : A_{g,(2,4)} \longrightarrow \mathbb{P}^N$

The theta constants provide a natural G-equivariant map

$$\Theta : A_{g,(2,4)} \longrightarrow \mathbb{P}^N, \qquad N+1 = 2^g.$$

Over $\mathbb{C}$, the map Θ is induced by the map

$$\mathbb{H}_g \longrightarrow \mathbb{P}^N, \qquad \tau \longmapsto (\ldots : \Theta[\sigma](\tau) : \ldots)_{\sigma \in (\mathbb{Z}/2\mathbb{Z})^g}$$

with theta constants

$$\Theta[\sigma](\tau) = \sum_{m \in \mathbb{Z}^g} e^{2\pi i^t(m+\sigma/2)\tau(m+\sigma/2)}.$$

$\Theta(A_{g,(2,4)})$ is birationally isomorphic with $A_{g,(2,4)}$.

For $g = 2$: $\Theta(A_{2,(2,4)}) = \mathbb{P}^3 - \{30 \text{ lines}\}$.

Determine $M_g \in \mathbf{Aut}(\mathbb{P}^N)$

Can easily find $M_g \in \mathrm{Aut}(\mathbb{P}^N)$ for $g \in (\mathbb{Z}/2\mathbb{Z})^{2g} \subset G$.
("Heisenberg group action")

For any transvection $t_v \in Sp(2g, \mathbb{F}_2)$ can find $M_{t_v} \in \mathrm{Aut}(\mathbb{P}^N)$
(M_{t_v} is a linear combination of I and M_v).

Hence can find M_g for any $g \in G$.

In case $g = 2,\ Sp(2g, \mathbb{F}_2) \cong S_6$ (symmetric group).
Transvections correspond to transpositions.

Can easily find element of order five $h \in G$ and corresponding $M_h \in \mathrm{Aut}(\mathbb{P}^3)$.

M_h has four fixed points in $\mathbb{P}^3 - \{\text{30 lines}\} = \Theta(A_{2,(2,4)})$,
By unicity, each fixed point is a $[(B_0, L_0, \alpha)]$ and $h = \phi^*$.

The eigenspaces $\mathbb{P}_\lambda$ in $\mathbb{P}^{15}$ of $M_h^{(k)}$

Recall: $(A_0, L, \phi_k) = (B_0 \times B_0, L_0 \boxtimes L_0, \phi_k := \phi \times \phi^k)$, $M_{\phi^*} = M_h$.

There are natural identifications:

$$\mathbb{P}^3 = \mathbb{P}\mathbb{C}^4, \quad \mathbb{P}^{15} = \mathbb{P}(\mathbb{C}^4 \otimes \mathbb{C}^4), \quad \Theta(A_0) = \Theta(B_0) \otimes \Theta(B_0).$$

$$M_h^{(k)} := M_{\phi_k^*} = M_{\phi^* \times (\phi^k)^*} = M_h \otimes M_h^k.$$

Can thus easily find the eigenspaces $\mathbb{P}_\lambda \subset \mathbb{P}^{15}$ of $M_h^{(k)}$ which contain $\Theta(A_0)$,

$$\Theta(A_0) \in \mathbb{P}_\lambda, \qquad \dim \mathbb{P}_\lambda = \begin{cases} 2 & k = 2, 3, \\ 3 & k = 4. \end{cases}$$

Hence $\Theta((A_{4,(2,4)})^{\phi_k^*})$ is of codimension one in $\mathbb{P}_\lambda$.

The equations for $\Theta(A_{g,(2,4)})$

Classical (even) theta constants ($\epsilon, \epsilon' \in (\mathbb{Z}/2\mathbb{Z})^g,\ \epsilon \cdot \epsilon' = 0$):

$$\theta[{\epsilon \atop \epsilon'}]^2 = \sum_{\sigma \in (\mathbb{Z}/2\mathbb{Z})^g} (-1)^{\sigma \cdot \epsilon'} \Theta[\sigma]\Theta[\sigma + \epsilon].$$

There are well-known relations between the even theta constants, for example:

$$\prod_{a,b \in (\mathbb{Z}/2\mathbb{Z})^2} \theta[{0000 \atop 00ab}] - \prod_{a,b} \theta[{0000 \atop 10ab}] - \prod_{a,b} \theta[{1000 \atop 00ab}] - \prod_{a,b} \theta[{1100 \atop 11ab}] = 0,$$

of the form $r_1 - r_2 - r_3 - r_4$. Get a relation between the squares and a polynomial F of degree 32:

$$0 = \prod_{\pm,\pm,\pm} (r_1 \pm r_2 \pm r_3 \pm r_4) = P(r_1^2, \ldots, r_4^2) = F(\ldots, \Theta[\sigma], \ldots).$$

The intersection $\Theta(A_{4,(2,4)}) \cap \mathbb{P}_\lambda$

Take two such polynomials F_1, F_2,
restrict them to the eigenspace $\mathbb{P}_\lambda$, find their GCD:

$$\overline{\Theta((A_{4,(2,4)})^{\phi_k^*})} = \begin{cases} \text{a conic in } \mathbb{P}^2 & k = 2, 3, \\ \text{a degree six surface in } \mathbb{P}^3 & k = 4. \end{cases}$$

Let $Q[{}^{\epsilon}_{\epsilon'}]$ be the quadric in $\mathbb{P}^{15}$ such that

$$Q[{}^{\epsilon}_{\epsilon'}] \cap \Theta(A_{4,(2,4)}) = \{\Theta(\tau) : \theta[{}^{\epsilon}_{\epsilon'}](\tau) = 0\}.$$

The boundary lies in at least $28 + 72 = 100$ such quadrics.
The conic lies inside $\Theta((A_{4,(2,4)})^{\phi_k^*})$
(so we have a **compact Shimura curve**). The surface meets the boundary in 5 points (**the cusps**).

Covers and the Schottky-Jung relation

$$\underbrace{A_{g,(2,4,8)} \longrightarrow A_{g,(4,8)} \longrightarrow A_{g,4} \longrightarrow A_{g,(2,4)}.}_{\text{group } (\mathbb{Z}/2\mathbb{Z})^M}$$

Intermediate 2:1 covers of $\Theta(A_{g,(2,4)})$ are given by

$$Y^2 = \sum_{\sigma\in(\mathbb{Z}/2\mathbb{Z})^g} (-1)^{\sigma\cdot\epsilon'} X_\sigma X_{\sigma+\epsilon} \qquad (\subset \mathbb{P}^{N+1}),$$

i.e. get **modular** covers branched over $Q[{\epsilon \atop \epsilon'}] \cap \Theta(A_{g,(2,4)})$.

The closure of the locus of **Jacobians** of genus 4 curves is:

$$\Theta(A_{g,(2,4)}) \cap (J=0), \quad J = 2^4 \sum \theta\left[{\epsilon \atop \epsilon'}\right]^{16} - \left(\sum \theta\left[{\epsilon \atop \epsilon'}\right]^8\right)^2,$$

viewed as polynomial of degree 16 in the X_σ $(= \Theta[\sigma])$.

Geometry of the Shimura curve

The Shimura curve lies inside the Jacobi locus.

It parametrizes the genus 4 curves with automorphism

$$C_\lambda : \quad y^5 = x(x-1)(x-\lambda), \qquad \psi(x,y) := (\zeta x, y),$$

the de Jong-Noot family. For $\lambda = 0, 1$ one has

$$J(C_\lambda) \cong B_0 \times B_0, \qquad \psi^* = (\phi^*, (\phi^2)^*) = \phi_2^*.$$

The Shimura curve lies in one of the quadrics $Q[{}^{\epsilon}_{\epsilon'}]$

(each C_λ has a 'vanishing even thetanull').

The remaining $136 - 1 = 135 = 5 \cdot 27$ quadrics intersect the curve in 12 points, corresponding to $B_0 \times B_0$ (with some level structure).

Parametrising the Shimura curve

The Shimura curve (a conic) is isomorphic to $\mathbb{P}^1$ and

$$\Theta((A_{4,(2,4)})^{\phi_2^*}) \cap (\cup' Q[^{\epsilon}_{\epsilon'}]) = \{0, \infty, \zeta^k, \alpha\zeta^k\}_{k=0,\dots,4},$$

where $\alpha = \zeta^3 + \zeta^2 + 1$.

Jacobians of some of the modular covers decompose into products of elliptic curves with $j \in \mathbb{Q}, \mathbb{Q}(\sqrt{5})$.

Among the corresponding modular forms is (a twist of) a Hilbert modular form of parallel weight two and conductor $8\sqrt{5}$.

The Mumford-Tate group

Another description of (A, L, ϕ):

- $A = V/\Lambda, \quad \Lambda \cong \mathbb{Z}[\zeta]^2, \quad V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$,
- $J : V \to V$ is the complex structure, $\quad J^2 = -I$,
- $c_1(L) = E : \Lambda \times \Lambda \to \mathbb{Z}, \qquad E(x, y) = trace({}^t x H \bar{y})$.
  H is skew Hermitian: ${}^t H = -\overline{H} \in M_2(\mathbb{Q}(\zeta))$.
- $\phi_* x = \zeta x$ for all $x \in V$.

Compatibility: $J \in SU(H)(\mathbb{R}), \quad SU(H) \cong D_1^\times$,
D is a quaternion algebra with center $F = \mathbb{Q}(\sqrt{5})$.
$F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R} \times \mathbb{R}, \qquad \sqrt{5} \longmapsto (\sqrt{5}, -\sqrt{5})$,

$$D_1^\times(\mathbb{R}) \cong SU(2) \times SU(1,1) \cong SU(2) \times SL(2, \mathbb{R}).$$

The Shimura curve is $\Gamma \backslash \mathbb{H}_1, \quad \Gamma \subset \mathrm{im}\,(D_1^\times(\mathbb{Z}) \to SL(2, \mathbb{R}))$.

Geometry of the Shimura surface

The Shimura surface has 5 cusps and has automorphism group S_5 (symmetric group). Equations (in a $\mathbb{P}^4$):

$$s_1 := x_1 + \ldots + x_5 \;=\; 0, \qquad s_2^3 + 10 s_3^2 - 20 s_2 s_4 \;=\; 0.$$

Singular points: 5 cusps (orbit of p_0, tgt cone: $xyz = 0$) and 24 nodes (orbit of q_0), corresponding to $B_0 \times B_0$:

$$p_0 \;:=\; (-4 : 1 : 1 : 1 : 1), \qquad q_0 \;:=\; (1 : \zeta : \zeta^2 : \zeta^3 : \zeta^4).$$

Locus of Jacobians: a curve of degree $6 \cdot 16 = 96$,

- a curve of degree 24 (no vanishing thetanull) parametrises $y^5 = x^3(x-1)^2(x-\lambda)$.
- 12 curves of degree three (and multipl. 2) parametrising $y^5 = x^4(x^2 + \lambda x + 1)$, hyperelliptic curves (in 10 $Q[{\epsilon \atop \epsilon'}]$'s), Weierstrass equation: $y^2 = (x^5 - 1)(x^5 - \mu_\lambda)$.

The canonical model of the Shimura surface

Canonical system: quadrics in $\mathbb{P}^3$ passing through the 5 cusps. There is an injective homomorphism $S_5 \hookrightarrow S_6$:

$$(1\,2) \longmapsto (1\,4)(2\,3)(5\,6), \quad (5\,4\,3\,2\,1) \longmapsto (2\,6\,5\,4\,3).$$

Equations for the canonical model of the Shimura surface (in a $\mathbb{P}^5$), a complete intersection of type $(3,3)$:

$$z_1 + z_2 + z_3 + z_4 + z_5 + z_6, \qquad z_1^3 + z_2^3 + z_3^3 + z_4^3 + z_5^3 + z_6^3,$$

and the following (alternating for S_5) cubic:

$$z_1z_2z_3 - z_1z_2z_4 - z_1z_2z_5 + \ldots + z_3z_5z_6 - z_4z_5z_6.$$

S_5 has a unique irreducible representation on $\mathbb{P}^5$, these are the 'unique' cubic invariants.

Arithmetic of the Shimura surface

Numerical invariants of the minimal model $\tilde{S}$ of the surface:

$$q = 0, \quad p_g = 5, \quad K^2 = 9, \quad \chi_{top} = 63, \quad h^{1,1} = 51, \quad \rho = 46.$$

The Hodge structure on H^2 splits:

$$H^2(\tilde{S}, \mathbb{Q}) \;=\; T \oplus_{\perp} S, \qquad \begin{cases} T = V^5, \;\; \dim\, V^{p,q} = 1, \; \forall (p,q), \\ S^{2,0} = S^{0,2}, \;\; \dim\, S^{1,1} = \rho. \end{cases}$$

The L-series of the Galois representation associated to V:

$$L(V, s) \stackrel{?}{=} L(\mathrm{Sym}^2 H^1(E), s), \qquad E: \;\; y^2 = x(x^2 + x - 1),$$

E is an elliptic curve with conductor 20, $j(E) = 16384/5$,
E is also a modular double cover of the Shimura curve.

The Shimura surface is a Hilbert modular surface

A Hilbert modular surface is the moduli space of abelian surfaces B such that

$$End(B) \otimes \mathbb{Q} \supset \mathbb{Q}(\sqrt{d}) \qquad d > 0.$$

It is obtained as (with $\mathbb{Q}(\sqrt{d}) \hookrightarrow \mathbb{R} \times \mathbb{R}$ as before):

$$\Gamma \backslash (\mathbb{H}_1 \times \mathbb{H}_1), \qquad \Gamma \subset SL_2(\mathbb{Q}(\sqrt{d})) \hookrightarrow SL_2(\mathbb{R}) \times SL_2(\mathbb{R}).$$

Any deformation of (A_0, L, ϕ_4) is isogeneous to B^2, for an abelian surface B with $\mathbb{Q}(\sqrt{5}) \subset End(B) \otimes \mathbb{Q}$.
Thus the Shimura surface is dominated by a **Hilbert modular surface**.

Mumford Tate group: $\quad SU(H) \cong SL_{2,\mathbb{Q}(\sqrt{5})}$.